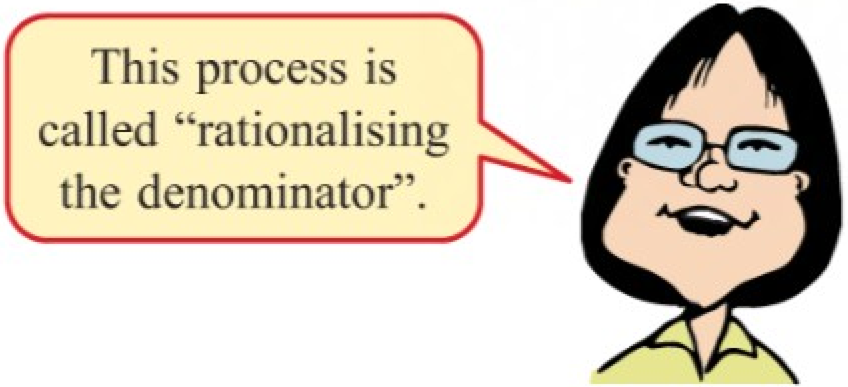
E DIVISION BY RADICALS

When an expression involves division by a radical, we can write the expression with an **integer denominator** which does **not** contain radicals.

For example, consider $\frac{6}{\sqrt{3}}$.

Since $\frac{\sqrt{3}}{\sqrt{3}} = 1$, we can multiply $\frac{6}{\sqrt{3}}$ by $\frac{\sqrt{3}}{\sqrt{3}}$ without changing its value.

$$\begin{aligned} \frac{6}{\sqrt{3}} &= \frac{6}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} \\ &= \frac{6 \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}} \\ &= \frac{6\sqrt{3}}{3} \qquad \{\text{since } \sqrt{a} \times \sqrt{a} = a\} \\ &= 2\sqrt{3} \end{aligned}$$


This process is called "rationalising the denominator".

Example 13 ◀)) **Self Tutor**

Express with integer denominator:

a $\dfrac{7}{\sqrt{3}}$

b $\dfrac{10}{\sqrt{5}}$

c $\dfrac{10}{2\sqrt{2}}$

Example 13 **Self Tutor**

Express with integer denominator:

a $\dfrac{7}{\sqrt{3}}$ **b** $\dfrac{10}{\sqrt{5}}$ **c** $\dfrac{10}{2\sqrt{2}}$

a

$$\begin{aligned}&\frac{7}{\sqrt{3}}\\ &= \frac{7}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}\\ &= \frac{7\sqrt{3}}{3}\end{aligned}$$

b

$$\begin{aligned}&\frac{10}{\sqrt{5}}\\ &= \frac{10}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}\\ &= \frac{10\sqrt{5}}{5}\\ &= 2\sqrt{5}\end{aligned}$$

c

$$\begin{aligned}&\frac{10}{2\sqrt{2}}\\ &= \frac{10}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}\\ &= \frac{10\sqrt{2}}{4}\\ &= \frac{5\sqrt{2}}{2}\end{aligned}$$

Now suppose the denominator has the form $a + b\sqrt{c}$.

The **radical conjugate** of $a + b\sqrt{c}$ is $a - b\sqrt{c}$.

The product of the radical conjugates is

$$\begin{aligned}(a + b\sqrt{c})(a - b\sqrt{c}) &= a^2 - (b\sqrt{c})^2 \\ &= a^2 - b^2 c\end{aligned}$$

which does not involve a radical.

So, given a denominator of the form $a + b\sqrt{c}$, we multiply the fraction by $\dfrac{a - b\sqrt{c}}{a - b\sqrt{c}}$.

Example 14

Self Tutor

Express $\frac{1}{3+5\sqrt{2}}$ with integer denominator.

Example 14 **Self Tutor**

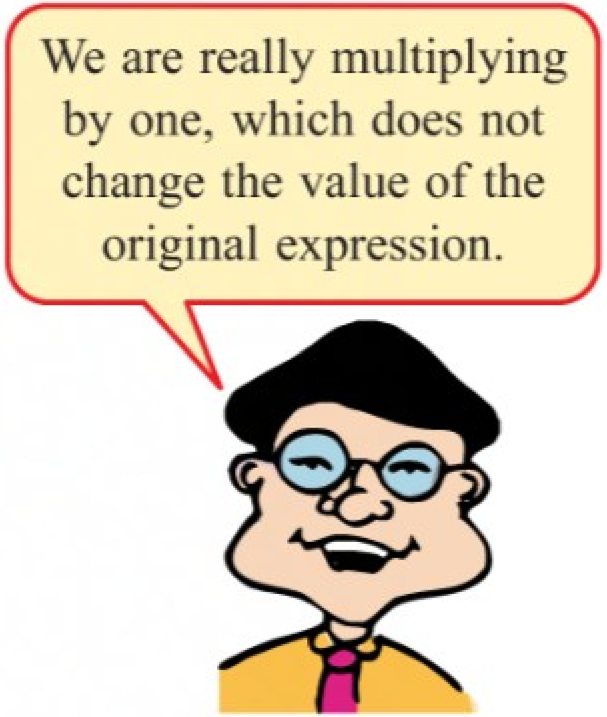
Express $\dfrac{1}{3+5\sqrt{2}}$ with integer denominator.

$$\frac{1}{3+5\sqrt{2}} = \left(\frac{1}{3+5\sqrt{2}}\right)\left(\frac{3-5\sqrt{2}}{3-5\sqrt{2}}\right)$$

$$= \frac{3-5\sqrt{2}}{3^2-(5\sqrt{2})^2} \qquad \{\text{using} \quad (a+b)(a-b)=a^2-b^2\}$$

$$= \frac{3-5\sqrt{2}}{9-50}$$

$$= \frac{5\sqrt{2}-3}{41}$$


We are really multiplying by one, which does not change the value of the original expression.

1 Express with integer denominator:

a $\frac{1}{\sqrt{2}}$
b $\frac{2}{\sqrt{2}}$
c $\frac{4}{\sqrt{2}}$
d $\frac{\sqrt{3}}{\sqrt{2}}$
e $\frac{\sqrt{7}}{3\sqrt{2}}$
f $\frac{1}{\sqrt{3}}$
g $\frac{3}{\sqrt{3}}$
h $\frac{4}{\sqrt{3}}$
i $\frac{\sqrt{7}}{\sqrt{3}}$
j $\frac{\sqrt{11}}{4\sqrt{3}}$
k $\frac{1}{\sqrt{5}}$
l $\frac{3}{\sqrt{5}}$
m $\frac{15}{\sqrt{5}}$
n $\frac{\sqrt{3}}{\sqrt{5}}$
o $\frac{125}{2\sqrt{5}}$
p $\frac{\sqrt{10}}{\sqrt{2}}$
q $\frac{1}{2\sqrt{3}}$
r $\frac{2\sqrt{2}}{\sqrt{3}}$
s $\frac{15}{2\sqrt{5}}$
t $\frac{1}{(\sqrt{2})^3}$

2 Express with integer denominator:

a $\frac{1}{3-\sqrt{5}}$ **b** $\frac{1}{2+\sqrt{3}}$ **c** $\frac{1}{4-\sqrt{11}}$ **d** $\frac{\sqrt{2}}{5+\sqrt{2}}$

e $\frac{\sqrt{3}}{3+\sqrt{3}}$ **f** $\frac{5}{2-3\sqrt{2}}$ **g** $\frac{-\sqrt{5}}{3+2\sqrt{5}}$ **h** $\frac{3-2\sqrt{7}}{2+3\sqrt{7}}$

3 Write in the form $a+b\sqrt{2}$ where $a, b \in \mathbb{Q}$:

a $\frac{4}{2-\sqrt{2}}$ **b** $\frac{-5}{1+\sqrt{2}}$ **c** $\frac{1-\sqrt{2}}{1+\sqrt{2}}$ **d** $\frac{\sqrt{2}-2}{3-\sqrt{2}}$

e $\frac{\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}}$ **f** $\frac{1+\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}}$ **g** $\frac{1}{1-\frac{\sqrt{2}}{3}}$ **h** $\frac{\frac{\sqrt{2}}{2}+1}{1-\frac{\sqrt{2}}{4}}$

Example 15

Self Tutor

Write $\dfrac{\sqrt{3}}{1-\sqrt{3}} - \dfrac{1-2\sqrt{3}}{1+\sqrt{3}}$ in simplest form.

Example 15

Self Tutor

Write $\dfrac{\sqrt{3}}{1-\sqrt{3}} - \dfrac{1-2\sqrt{3}}{1+\sqrt{3}}$ in simplest form.

$$\begin{aligned}
\frac{\sqrt{3}}{1-\sqrt{3}} - \frac{1-2\sqrt{3}}{1+\sqrt{3}} &= \left(\frac{\sqrt{3}}{1-\sqrt{3}}\right)\left(\frac{1+\sqrt{3}}{1+\sqrt{3}}\right) - \left(\frac{1-2\sqrt{3}}{1+\sqrt{3}}\right)\left(\frac{1-\sqrt{3}}{1-\sqrt{3}}\right) \\
&= \frac{\sqrt{3}+3}{1-3} - \frac{1-\sqrt{3}-2\sqrt{3}+6}{1-3} \\
&= \frac{\sqrt{3}+3}{-2} - \frac{7-3\sqrt{3}}{-2} \\
&= \frac{\sqrt{3}+3-7+3\sqrt{3}}{-2} \\
&= \frac{-4+4\sqrt{3}}{-2} \\
&= \frac{-4}{-2} + \frac{4\sqrt{3}}{-2} \\
&= 2-2\sqrt{3}
\end{aligned}$$

4 Write in simplest form:

a $\dfrac{1+\sqrt{2}}{1-\sqrt{2}}+\dfrac{1-\sqrt{2}}{1+\sqrt{2}}$ **b** $\dfrac{2+\sqrt{5}}{2-\sqrt{5}}-\dfrac{\sqrt{5}}{2+\sqrt{5}}$ **c** $\dfrac{4-\sqrt{3}}{3-2\sqrt{2}}-\dfrac{2\sqrt{3}}{3+2\sqrt{2}}$

5 **a** Suppose a and b are positive integers. Show that $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})$ is also an integer.

b Write with an integer denominator:

i $\dfrac{4}{\sqrt{7}+\sqrt{2}}$ **ii** $\dfrac{2\sqrt{5}}{\sqrt{5}-\sqrt{2}}$ **iii** $\dfrac{\sqrt{3}+2\sqrt{13}}{\sqrt{3}-\sqrt{13}}$

6 Write $\sqrt{\dfrac{3+2\sqrt{2}}{3-2\sqrt{2}}}$ in the form $a+b\sqrt{2}$ where $a, b \in \mathbb{Q}$.

7 If $\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{2}} = p$, find $\sqrt{6}$ in terms of p.

8 If $x = \sqrt{5} - \sqrt{3}$, find x^2 and hence show that $x^4 - 16x^2 + 4 = 0$.
You have just shown that one solution of $x^4 - 16x^2 + 4 = 0$ is $x = \sqrt{5} - \sqrt{3}$.

9 Suppose $u_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$. Evaluate u_n for $n = 1, 2, 3, 4, 5$, and 6.

1 **a** $\frac{\sqrt{2}}{2}$ **b** $\sqrt{2}$ **c** $2\sqrt{2}$ **d** $\frac{\sqrt{6}}{2}$ **e** $\frac{\sqrt{14}}{6}$
f $\frac{\sqrt{3}}{3}$ **g** $\sqrt{3}$ **h** $\frac{4\sqrt{3}}{3}$ **i** $\frac{\sqrt{21}}{3}$ **j** $\frac{\sqrt{33}}{12}$
k $\frac{\sqrt{5}}{5}$ **l** $\frac{3\sqrt{5}}{5}$ **m** $3\sqrt{5}$ **n** $\frac{\sqrt{15}}{5}$ **o** $\frac{25\sqrt{5}}{2}$
p $\sqrt{5}$ **q** $\frac{\sqrt{3}}{6}$ **r** $\frac{2\sqrt{6}}{3}$ **s** $\frac{3\sqrt{5}}{2}$ **t** $\frac{\sqrt{2}}{4}$

2 **a** $\dfrac{3+\sqrt{5}}{4}$ **b** $\dfrac{2-\sqrt{3}}{1}$ **c** $\dfrac{4+\sqrt{11}}{5}$
d $\dfrac{5\sqrt{2}-2}{23}$ **e** $\dfrac{\sqrt{3}-1}{2}$ **f** $\dfrac{10+15\sqrt{2}}{-14}$
g $\dfrac{3\sqrt{5}-10}{11}$ **h** $\dfrac{-48+13\sqrt{7}}{59}$

3 **a** $4+2\sqrt{2}$ **b** $5-5\sqrt{2}$ **c** $-3+2\sqrt{2}$
d $-\frac{4}{7}+\frac{1}{7}\sqrt{2}$ **e** $1+\sqrt{2}$ **f** $3+2\sqrt{2}$
g $\frac{9}{7}+\frac{3}{7}\sqrt{2}$ **h** $\frac{10}{7}+\frac{6}{7}\sqrt{2}$

4 **a** -6 **b** $-14-2\sqrt{5}$ **c** $12+8\sqrt{2}-9\sqrt{3}+2\sqrt{6}$

5 **a** $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})$
$= (\sqrt{a})^2-(\sqrt{b})^2$ {difference of two squares}
$= a-b$ which is an integer since $a, b \in \mathbb{Z}^+$

b **i** $\dfrac{4\sqrt{7}-4\sqrt{2}}{5}$ **ii** $\dfrac{10+2\sqrt{10}}{3}$ **iii** $\dfrac{-29-3\sqrt{39}}{10}$

6 $3+2\sqrt{2}$ **7** $\sqrt{6}=\dfrac{\sqrt{2}-\sqrt{3}}{p}$ $\left(or\quad \sqrt{6}=\dfrac{12}{5-6p^2}\right)$

8 $x^2=8-2\sqrt{15}$ and so $2\sqrt{15}=8-x^2$
Now square both sides.

9 $u_1=1,\ u_2=1,\ u_3=2,\ u_4=3,\ u_5=5,\ u_6=8$